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EMBEDDINGS OF AND INTO NERODE SEMIRINGS

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ABSTRACT

Continuing the study of recursive ultrapowers launched by Hirschfeld in 1975 ([H]), we begin to investigate the more detailed embedding properties of these structures. Some related results in isol theory are noted.

1. Introduction and preliminaries

This paper is a continuation of work begun in [McL$_2$] and [McL$_3$]. We are motivated by the question of how the study of certain subsemirings of Λ, the semiring of all isols, fits in with the arithmetical model theory discussed in Chapters 8–13 of [H–W]. (Relative to [McL$_2$] and [McL$_3$], this is a matter of hindsight: we were not familiar with [H–W] at the time [McL$_2$] was written, and had already completed the main text of [McL$_3$] before noticing that [H–W] provides a good general frame of reference for our inquiries about semirings of isols.)

Several of the results to be obtained here are negative in character. Thus, while any countable model of full Π_2^0 arithmetic is embeddable in some Nerode semiring, we shall prove that a Nerode semiring will not *cofinally* embed any nonstandard model $\mathcal{M}$ of Π_2^0 arithmetic such that $\mathcal{M}$ satisfies all 4-quantifier theorems of (first-order) Peano Arithmetic. Further, we shall verify that while any Nerode semiring can be embedded in a countable model of true arithmetic, no Nerode semiring can be end-extended to a model of the 3-quantifier theorems of Peano Arithmetic. In §4, we note a general fact about unions of "$\forall\exists$-correct" semirings of isols that implies the nonclosure of the set of so-called *RST isols* under both addition and multiplication.

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The origins of our topic lie in the 1959 abstract [F–S–T], which announced that among all homomorphic images of the semiring **R** of recursive functions, only ω is a model of Peano Arithmetic. The most interesting of such homomorphic images, other than ω, are the so-called "recursive ultrapowers". These are obtained by restricting the usual ultrapower definition to the class of recursive functions and to the (nonprincipal) ultrafilters of the boolean algebra of recursive sets. In 1966, Nerode showed in [N_3] that, to within isomorphism, the recursive ultrapowers are identical with certain countable semirings of isols. Then in 1975, Hirschfeld showed in [H] that the recursive ultrapowers satisfy a Σ_1^0 version of Los' Theorem, that ω is definable, in a uniform way, in each of them, and that they are models of two-quantifier arithmetic in one or another of which *any* model of two-quantifier arithmetic can be embedded. The latter embedding result provided the link between isols and arithmetical model theory that is the focus of our interest.

In matters of notation and terminology, we shall follow standard recursion-theoretic and model-theoretic usages wherever possible. The remainder of this section is devoted to the cataloguing of certain conventions and background observations.

Let $\mathscr{L}$ be the language for first-order arithmetic whose non-logical constants are $+, \cdot, <, \bar{0}, \bar{1}, \bar{2}, \ldots$. Let *"PA"* denote the usual axiom system for Peano Arithmetic based on the language $\mathscr{L}$, as specified in [P] (adjusted, of course, for our choice of primitive symbols of $\mathscr{L}$); thus, in the notation of [P], $PA = P^- + \bigcup_{n=0}^{\infty} I\Sigma_n$. If $\mathscr{M}$ is an $\mathscr{L}$-structure, let "$|\mathscr{M}|$" denote the universe of $\mathscr{M}$.

In §2, before entering into the main considerations of the paper, we shall pause to examine one feature of the notion of $\mathscr{M}$-recursive function that we here choose to adopt. Our notion is not necessarily the "usual" one, and is formulated as follows:

Let $\psi(\boldsymbol{x}, y)$ be a Σ_1^0 formula of $\mathscr{L}$; i.e., $\psi(\boldsymbol{x}, y)$ is a formula consisting of an initial (possibly empty) block of existential quantifiers followed by a "bounded kernel" in which every quantifier Q is of the form $\exists w \leqq t$ or $\forall w \leqq t$, t being a term of $\mathscr{L}$. (It is understood, of course, that the notation "$\psi(\boldsymbol{x}, y)$" dictates that exactly the variables $\boldsymbol{x}(=x_1, \ldots, x_k)$ and y have free occurrences in ψ.) Suppose further that $\psi(\boldsymbol{x}, y)$ defines a function $y = f(\boldsymbol{x})$ *in* ω, $\omega =$ the standard natural number system. If ω is an initial substructure of an $\mathscr{L}$-structure $\mathscr{M}$ (i.e., if $\mathscr{M}$ is an end-extension of ω with respect to $<$), and if $\psi(\boldsymbol{x}, y)$ defines *in* $\mathscr{M}$ a function $g: |\mathscr{M}|^k \to |\mathscr{M}|$ such that $g\restriction_{\omega^k} = f$, then we say that g is *$\mathscr{M}$-recursive*. Conversely, no function $g: |\mathscr{M}|^k \to |\mathscr{M}|$ shall be counted as $\mathscr{M}$-recursive unless it so arises from some Σ_1^0 formula $\psi(\boldsymbol{x}, y)$ of $\mathscr{L}$.

(In particular, we do not define the notion of $\mathcal{M}$-recursiveness, at least for present purposes, unless $\mathcal{M}$ is an end-extension of ω.)

Let $\mathcal{L}'$ be the language obtained from $\mathcal{L}$ by adjoining a binary function symbol "(,)" whose interpretation, in ω, is that $z = (x, y)$ holds $\Leftrightarrow p_y^z \mid x$ but $p_y^{z+1} \nmid x$; here p_y is the y-th prime in order of magnitude. As is customary, we write "$z = (x)_y$" in place of "$z = (x, y)$." The function $z = (x)_y$ is primitive recursive (see, e.g., [M, Chapter 3]): hence, by [P, Fact 10], $z = (x)_y$ is defined in ω by a Σ_1^0 formula $\phi(x, y, z)$ such that $PA \vdash \forall x \forall y \exists ! z \phi(x, y, z)$. Moreover, it is known (see, for instance, [G]) that each instance of the Davis–Putnam–Robinson–Matijasevic Theorem (DPRM) on diophantine representation is *provable in PA*, for a suitable choice of the diophantine representing predicate. So, as in [G, pp. 135–6], the above formula ϕ can be assumed to be a diophantine predicate; i.e., we may take ϕ to be of the form $\exists \mathbf{w} D(\mathbf{w}, x, y, z)$, where $D(\mathbf{w}, x, y, z)$ is an equation between polynomials with coefficients in ω. Thus, we are choosing not just *any* diophantine predicate which happens to be equivalent to ϕ in ω, but are requiring in fact that $PA \vdash \forall x \forall y \forall z [\phi(x, y, z) \leftrightarrow \exists \mathbf{w} D(\mathbf{w}, x, y, z)]$. Since $\exists \mathbf{w} D(\mathbf{w}, x, y, z)$ is purely existential, it is upwardly persistent for $\mathcal{L}$-structures. This means that if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $a, b, c \in \mathcal{M}_1$ and $\mathcal{M}_1 \models \exists \mathbf{w} D(\mathbf{w}, a, b, c)$, then $\mathcal{M}_2 \models \exists \mathbf{w} D(\mathbf{w}, a, b, c)$. (We shall always use the notation "$\mathcal{M}_1 \subseteq \mathcal{M}_2$" to mean that $\mathcal{M}_1$ *is a substructure of* $\mathcal{M}_2$.) Putting these remarks together, we see that if $\mathcal{M}$ is any model of PA (or of ω-true Π_2^0 arithmetic) then $\exists \mathbf{w} D(\mathbf{w}, x, y, z)$ determines an $\mathcal{M}$-recursive function, and $\mathcal{M}$ can be viewed as an $\mathcal{L}'$-structure via the "defining axiom" π_D, where $\pi_D = \forall x \forall y \forall z [z = (x)_y \leftrightarrow \exists \mathbf{w} D(\mathbf{w}, x, y, z)]$. We shall continue to use the symbol "$(x)_y$" to refer to this $\mathcal{M}$-recursive function, letting context determine which $\mathcal{M}$ we have reference to. We denote by "$PA^{\mathcal{L}'_D}$" the axiom system $PA \cup \{\pi_D\}$.

1.1. Definition. Let $\mathcal{M}$ be a structure for the language $\mathcal{L}$ (for the language $\mathcal{L}'$), and let k be a positive integer. We say that $\mathcal{M} \models PA_k$ ($\mathcal{M} \models PA_k^{\mathcal{L}'_D}$), or that $\mathcal{M}$ is a *model of* PA_k (*of* $PA_k^{\mathcal{L}'_D}$), in case $\mathcal{M} \models \phi$ holds for every Π_k^0 sentence ϕ of $\mathcal{L}$ (of $\mathcal{L}'$) such that $PA \vdash \phi$ (such that $PA^{\mathcal{L}'_D} \vdash \phi$). (Naturally $\mathcal{M}$ then also satisfies τ for every sentence τ that is *logically implied by* such a ϕ.)

In §3 in particular, we shall want to be able to work more or less interchangeably with $\mathcal{L}$ and $\mathcal{L}'$; and we shall there be concerned especially with Π_4^0 sentences. Hence, we observe the following proposition. (In view of the conservative nature of definitional extensions, this proposition in all likelihood does not really need the individual and detailed attention we are about to give it; however, we take the opportunity to provide a sampling of the formal arithmetical devices to be used throughout the paper.)

1.2. PROPOSITION. *Let* $\phi = \forall x \exists y \forall u \exists v \psi(x, y, u, v)$ *be a* Π_4^0 *sentence of* $\mathscr{L}'$ *such that* $PA^{\mathscr{L}'_D} \vdash \phi$. *Suppose* $\mathscr{M}$ *is an* $\mathscr{L}$*-structure such that* $\mathscr{M} \models \tau$ *for every* Π_4^0 *sentence* τ *of* $\mathscr{L}$ *for which* $PA \vdash \tau$. *Then* $\mathscr{M}$, *viewed as an* $\mathscr{L}'$*-structure via* π_D, *satisfies* ϕ. (*Thus*, $\mathscr{M} \models PA_4 \Leftrightarrow \mathscr{M} \models PA_4^{\mathscr{L}'_D}$.)

PROOF. Let us apply to ϕ the "translation algorithm" indicated by Gaifman on p. 137 of [G]. Doing so we obtain, for suitable choice of a quantifier-free $\mathscr{L}$-formula ζ and for a ψ-determined (not necessarily homogeneous) string Q of *bounded* quantifiers, the following equivalence that is true in $\mathscr{M}$ when $\mathscr{M}$ is viewed, via π_D, as an $\mathscr{L}'$-structure, and is also true in any model of $PA^{\mathscr{L}'_D}$: $\phi \leftrightarrow \tau'$, where τ' is the sentence

$$\forall x \exists y \forall u \exists v Qz \exists w \zeta(x, y, u, v, z, w).$$

Now, by iterated application of the "Collection Schema" ([P, p. 2]), working "from the inside out," we see that $PA \vdash \xi$, where ξ is the sentence

$$\forall x \forall y \forall u \forall v [Qz \exists w \zeta(x, y, u, v, z, w) \leftrightarrow \exists t Q_0 s \zeta'(x, y, u, v, t, s)]$$

with $Q_0 s$ = a string of bounded quantifiers and ζ' = a quantifier-free formula of $\mathscr{L}$. Moreover, all the instances of the Collection Schema that are needed for this are (in the terminology of [P]) instances of $B\Sigma_1$ and hence belong to PA_4; and all the corresponding right-to-left implications are also in PA_4 (indeed, PA_3). Hence $\mathscr{M} \models \xi$. It follows that $\mathscr{M} \models \tau' \leftrightarrow \tau$, where τ is the $\mathscr{L}$-sentence

$$\forall x \exists y \forall u \exists v \exists t Q_0 s \zeta'(x, y, u, v, t, s).$$

Now, the $\mathscr{L}'$-structures that are models of $PA^{\mathscr{L}'_D}$ are precisely the expansions to $\mathscr{L}'$, via π_D, of the $\mathscr{L}$-structures that satisfy PA; conversely, the $\mathscr{L}$-structures that are models of PA are precisely the $\mathscr{L}$-reducts of those $\mathscr{L}'$-structures $\mathscr{M}'$ such that $\mathscr{M}' \models PA^{\mathscr{L}'_D}$. Hence any model of $PA^{\mathscr{L}'_D}$ satisfies $\tau' \leftrightarrow \tau$, and therefore satisfies $\phi \leftrightarrow \tau$. Since $PA^{\mathscr{L}'_D} \vdash \phi$, this implies that if $\mathscr{M}'$ is any model of $PA^{\mathscr{L}'_D}$, then $\mathscr{M}' \models \tau$. It follows that τ holds in every model of PA, and so $PA \vdash \tau$. τ, however, is Π_4^0; so, by our hypothesis on $\mathscr{M}$, $\mathscr{M} \models \tau$. Hence $\mathscr{M} \models \tau'$. Thus $\mathscr{M}$, viewed via π_D as an $\mathscr{L}'$-structure, satisfies τ' and so satisfies ϕ.

2. $\mathcal{M}$-recursiveness of functions

Before turning to our main concerns, i.e., the embedding and extension considerations of §§3–5, we think it worthwhile to discuss in detail one aspect of "$\mathcal{M}$-recursiveness," for functions acting on $\mathcal{L}$-structures $\mathcal{M}$.

Given two $\mathcal{L}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\omega \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2$, with ω being an *initial* substructure of both $\mathcal{M}_1$ and $\mathcal{M}_2$, we wish to consider with some care the following question: What additional conditions, if any, on the pair $(\mathcal{M}_1, \mathcal{M}_2)$ are needed to ensure that the $\mathcal{M}_1$-recursive functions are just the restrictions to $\mathcal{M}_1$ of the $\mathcal{M}_2$-recursive functions? We shall not attempt to give a definitive answer to this question, but will content ourselves with showing that a measure of caution is in order relative to assuming, for a given pair $(\mathcal{M}_1, \mathcal{M}_2)$ of end-extensions of ω with $\mathcal{M}_1 \subseteq \mathcal{M}_2$, that f is $\mathcal{M}_1$-recursive $\Leftrightarrow (\exists g)$ [g is $\mathcal{M}_2$-recursive and $f = g\lceil_{\mathcal{M}_1}$].

In particular, we will see that it does *not* suffice to assume merely that $\mathcal{M}_1$ and $\mathcal{M}_2$ are both models of PA. (This represents a divergence, at low quantifier level, between our present notion of $\mathcal{M}$-recursiveness and the one adopted, e.g., in [G].)

2.1. Definition. Let $\mathcal{M} \models PA$. We shall say that $\mathcal{M}$ is *0-correct* if $\mathcal{M} \models \phi$ for every bounded sentence ϕ of $\mathcal{L}'$ such that $\omega \models \phi$. We say that $\mathcal{M}$ is *n-correct*, $n \geqq 1$, if $\mathcal{M} \models \phi$ for every ω-true Π^0_n sentence ϕ of $\mathcal{L}'$. Call $\mathcal{M}$ *strictly n-correct* if $\mathcal{M}$ is n-correct but there is a Π^0_{n+1} sentence ψ *of* $\mathcal{L}$ such that $\mathcal{M} \not\models \psi$. Finally, let $\mathcal{M}$ be called *correct* if it is n-correct for all $n \in \omega$. (Thus, $\mathcal{M}$ is *correct* just in case it is a model of ω-true arithmetic.)

To arrive at the result (Theorem 2.4) that is our objective in this section, we shall have recourse to two lemmas (the first of which actually interests us, in the present context, only in the case $n = 1$).

2.2. Lemma. (i) *For each $n \geqq 0$, there exists a strictly n-correct countable model of PA.*

(ii) *If $\mathcal{M}$ is a strictly n-correct model of PA, $n \geqq 0$, then the counterexample to $(n+1)$-correctness of $\mathcal{M}$ can be taken to be an $\forall \exists \cdots Q_{n+1}$ prenex normal-form sentence of $\mathcal{L}$, i.e., a sentence ϕ of $\mathcal{L}$ consisting of $n+1$ alternating homogeneous blocks of unbounded quantifiers followed by a quantifier-free formula.*

Proof. Part (i) of the lemma is just an elaborated version of the basic Gödel Incompleteness Theorem, and is well known; accordingly, we omit proof. To verify part (ii), suppose that $\mathcal{M}$ is strictly n-correct, and that ψ is a Π^0_{n+1} sentence of $\mathcal{L}$ witnessing the failure of $\mathcal{M}$ to be $(n+1)$-correct. Let ψ be

$$\forall \boldsymbol{u}\, \exists \boldsymbol{v} \cdots Q_{n+1} \boldsymbol{w} \xi(\boldsymbol{u}, \boldsymbol{v}, \ldots, \boldsymbol{w}),$$

where $\xi(\boldsymbol{u}, \boldsymbol{v}, \ldots, \boldsymbol{w})$ is a bounded formula of $\mathcal{L}$. Suppose first that $n+1$ is even, so that $Q_{n+1} = \exists$. Then, since *PA* proves each instance of *DPRM*, there exist polynomials $P_1(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_j)$, with coefficients in ω, such that the equivalence

$$\forall \boldsymbol{u}\, \forall \boldsymbol{v} \cdots [Q_{n+1} \boldsymbol{w} \xi(\boldsymbol{u}, \boldsymbol{v}, \ldots, \boldsymbol{w}) \leftrightarrow \exists y_1 \cdots \exists y_j [P_1(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots y_j) = P_2(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_j))]$$

is provable in *PA* and hence true in $\mathcal{M}$. It follows that the sentence

$$\forall \boldsymbol{u}\, \exists \boldsymbol{v} \cdots \exists y_1 \cdots \exists y_j [P_1(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_j) = P_2(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_j)]$$

witnesses the failure of $\mathcal{M}$ to be $(n+1)$-correct. In a similar way, if $n+1$ is odd, and therefore $Q_{n+1} = \forall$, we can conclude that there are polynomials $R_1(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_l)$ and $R_2(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_l)$ with natural number coefficients such that the sentence

$$\forall \boldsymbol{u}\, \exists \boldsymbol{v} \cdots \forall y_1 \cdots \forall y_l [R_1(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_l) \neq R_2(\boldsymbol{u}, \boldsymbol{v}, \ldots, y_1, \ldots, y_l)]$$

witnesses the failure of $\mathcal{M}$ to be $(n+1)$-correct.

(As a matter of fact, we shall not need part (ii) of Lemma 2.2 in our proof of Theorem 2.4; we simply thought it worth mentioning.)

2.3. LEMMA. *For each countable* 1*-correct model* $\mathcal{M}$ *of PA, there is a countable correct model* $\mathcal{M}'$ *such that* $\mathcal{M} \subseteq \mathcal{M}'$.

PROOF. Since $\mathcal{M}$ is 1-correct, it is Diophantine correct in the sense of [N_3]. Therefore, by [N_3, Th. 5.2], the additive-multiplicative structure of $\mathcal{M}$ is embeddable in the isols. Let S be the image of $\mathcal{M}$ in Λ under such an embedding. Now, $\mathcal{M}$, and therefore also S, satisfies all universal statements that are true in the ordinary additive and multiplicative arithmetic of ω. Hence S generates an integral domain within the ring Λ^* of isolic integers. (For a description of Λ^*, see [N_2] or [McL_1, Ch. 13].) Therefore, by [N_3, Th. 5.1] (and its proof), S is embeddable in a countable correct model $\mathcal{M}'$. Consequently $\mathcal{M}$ itself is so embeddable, and the lemma is proved. (Note that since $\mathcal{M} \models PA$, S is linearly ordered in accordance with the condition $X \leqq Y \Leftrightarrow \exists Z[X + Z = Y]$, which is the defining condition for $\leqq$ in Λ; thus, each of the embeddings just mentioned respects the order relation and so in fact pertains to the full language $\mathscr{L}$.)

Now to the main point of this section, Theorem 2.4. We do not know whether 2.4(ii) is, in contrast to 2.4(i), actually a novel observation. It would certainly not surprise us to learn that it is already present, in some form, somewhere in the literature; but we are not aware of such an occurrence. Theorem 2.4(ii) shows, on the one hand, that a larger model $\mathscr{M}_2$ of PA may admit more "recursive" functions than some smaller model $\mathscr{M}_1$ can accommodate via restriction; on the other hand, it also follows from 2.4(ii) that a smaller model may admit recursive functions that are not extendible, no matter what Σ_1^0 defining predicate is used, to the larger model. Indeed, our proof of Theorem 2.4 will exhibit these converse deficiences in a single chain $\omega \subseteq \mathscr{M}_1 \subseteq \mathscr{M}_2$ of models of PA.

2.4. THEOREM. (i) (Hirschfeld) *If $\mathscr{M}_1$ and $\mathscr{M}_2$ are models of ω-true Π_2^0 arithmetic such that $\mathscr{M}_1 \subseteq \mathscr{M}_2$, then* $\forall f[f$ is $\mathscr{M}_1$-recursive $\leftrightarrow \exists g$ *(g is $\mathscr{M}_2$-recursive) and* $f = g\lceil_{\mathscr{M}_1}]$.

(ii) *There exists a triple $(\mathscr{M}_1, \mathscr{M}_2, r)$ such that $\mathscr{M}_1 \models PA$, $\mathscr{M}_2 \models PA$, $\mathscr{M}_1 \subseteq \mathscr{M}_2$, $r: \omega^k \to \omega$ is for some k an ordinary k-place recursive function, and the following assertions hold:*

(iia) *every Σ_1^0 formula $\phi(\boldsymbol{x}, y)$ of $\mathscr{L}$ that defines a recursive function in ω determines an $\mathscr{M}_2$-recursive function;*

(iib) *no Σ_1^0 formula $\phi(\boldsymbol{x}, y)$ of $\mathscr{L}$ that defines r in ω determines an $\mathscr{M}_1$-recursive function.*

PROOF. Theorem 2.4(i) is, as is implicit in [H], a rather routine consequence of $DPRM$. To prove it, let $\phi(\boldsymbol{x}, y)$ be any Σ_1^0 formula of $\mathscr{L}$ that defines a function in ω. Then, as noted in [H], there is an ω-true Π_2^0 sentence ϕ_1 stating that $\phi(\boldsymbol{x}, y)$ defines a function in ω and another Π_2^0 sentence ϕ_2 expressing the equivalence (in ω) of $\phi(\boldsymbol{x}, y)$ with a suitably chosen diophantine predicate $\exists \boldsymbol{w}[P(\boldsymbol{w}, \boldsymbol{x}, y) = Q(\boldsymbol{w}, \boldsymbol{x}, y)]$. Then we have $\mathscr{M}_i \models \phi_1 \wedge \phi_2$, $i = 1, 2$. Putting that fact together with the upward persistence of existential predicates, we get Theorem 2.4(i).

To prove Theorem 2.4(ii), we begin by invoking Lemma 2.2(i): let $\mathscr{M}_1$ be a strictly 1-correct countable model of PA. Applying Lemma 2.3 to $\mathscr{M}_1$, let $\mathscr{M}_2$ be a countable correct model such that $\mathscr{M}_1 \subseteq \mathscr{M}_2$. Since $\mathscr{M}_2$ is correct, every Σ_1^0 formula of $\mathscr{L}$ that defines a recursive function in ω also defines one in $\mathscr{M}_2$. Letting f be an arbitrary j-place $\mathscr{M}_2$-recursive function, consider *any* Σ_1^0 formula $\phi(\boldsymbol{x}, y)$ such that $\phi(\boldsymbol{x}, y)$ defines $f\lceil_{\omega^j}$ in ω. Then *provably in PA*, there are polynomials $P(w, x, y)$ and $Q(w, x, y)$ such that $\omega \models \psi$ where $\psi = \forall x \forall y[\phi(x, y) \leftrightarrow \exists w(P(w, x, y) = Q(w, x, y))]$. That is to say, $PA \vdash \psi$. But then ψ holds in each of $\mathcal{M}_1$, $\mathcal{M}_2$, as well as in ω. Hence (on account of the upward persistence of existential predicates), if $\phi(x, y)$ defines in $\mathcal{M}_1$ an $\mathcal{M}_1$-recursive function g then $g = f \restriction_{\mathcal{M}_1}$. Now, $\mathcal{M}_2$ is a model of ω-true Π_2^0 arithmetic whereas $\mathcal{M}_1$ *is not*; so, by [H, Theorem 1.8], we can choose f to be an $\mathcal{M}_2$-recursive function of k arguments, for some k, such that $\mathcal{M}_1$ is not closed under the action of f. For such such a choice of f, it follows from our previous remarks that if $\phi(x, y)$ is *any* Σ_1^0 formula that defines $f \restriction_{\omega^k}$ in ω then $\phi(x, y)$ *does not define an* $\mathcal{M}_1$*-recursive function*. Taking $r = f$, the proof of Theorem 2.4(ii) is complete.

COMMENTS. In [G], the notion of $\mathcal{M}$-recursiveness employed seems to be that obtained by restricting the Σ_1^0 formulas in our definition to those which are also *PA-provably equivalent to* Π_1^0 *formulae*, i.e., in the notation of [P], to those which are "$\Delta_1(P)$." Thus, Gaifman gets a nice restriction–extension property, for *all* models of *PA*.

Theorem 2.4(ii) will have no impact on our work in §§3 and 4, since the uses we make of $\mathcal{M}$-recursive functions in those sections will be subject to the asumption that $\mathcal{M}$ is a model of full Π_2^0 arithmetic. In §5, we shall have need of one additional (somewhat minor) consideration of the kind we have been dealing with in the present section. That consideration will have to do with *partial* $\mathcal{M}$-recursive functions, and we shall defer it to §5 where it is needed.

Finally, it is natural to ask whether we can take $k = 1$ in choosing the function r of Theorem 2.4(ii). The answer is: yes, we can. This is so because $\mathcal{M}_1 \models PA$. The function $f(x_0, \dots, x_{k-1}) = 2^{x_0} \cdot \dots \cdot p_{k-1}^{x_{k-1}}$ is *provably* recursive, and therefore defines an $\mathcal{M}_1$-recursive function. It follows from this, via consideration of the definition schema

$$\psi(z, y) \leftrightarrow \exists x_0 \leqq z \cdots \exists x_{k-1} \leqq z[(z = 0 \wedge y = 0)$$

$$\vee (2^{x_0} \mid z \wedge 2^{x_0+1} \nmid z \wedge \cdots \wedge p_{k-1}^{x_{k-1}} \mid z \wedge p_{k-1}^{x_{k-1}+1} \nmid z \wedge \phi(x_0, \dots, x_{k-1}, y))]$$

that if $\mathcal{M}_1$ were closed under all 1*-place* $\mathcal{M}_2$-recursive functions, it would be closed under *all* $\mathcal{M}_2$-recursive functions.

3. Confinal embeddings into Nerode semirings

Recall ([McL$_2$, McL$_3$]) that a countable semiring **R** of isols is called a *Nerode semiring* if it has the form $\{f_\Lambda(X) \mid f: \omega \to \omega \text{ is recursive}\}$ for some fixed infinite isol X (a so-called "*RST* isol") such that $X \in \mathrm{dom}(f_\Lambda)$ for all recursive f; here f_Λ denotes the Myhill–Nerode extension ([N$_1$]; see also [McL$_1$, Ch. 11–12]) of f to the isols. These semirings were first studied in [N$_3$], where it was shown that they are isomorphically identifiable with the so-called (non-trivial) *recursive ultrapowers*. They are therefore models of ω-true Π_2^0 arithmetic ([H] and [N$_3$]; see the discussion in [McL$_2$]); hence, by [McL$_3$, Lemma 2], the **R**-recursive functions, $\mathbf{R}$ = a Nerode semiring, are just the functions $f_\Lambda \restriction_{\mathbf{R}^k}$ where $f: \omega^k \to \omega$ is recursive. As in [McL$_3$], we shall use "N(X)" as notation for the Nerode semiring $\{f_\Lambda(X) \mid f \text{ recursive}\}$.

Certain special Nerode semirings, the "tame models" (see [B], [McL$_2$], or [McL$_3$]), are "cofinal embedders", in the sense that any nonstandard 2-correct substructure is a *cofinal* substructure ([McL$_3$, Th. 3]). In fact, it turns out that these "tame models" are actually minimal; i.e., they have no proper non-standard 2-correct substructures. (We are indebted to Erik Ellentuck for pointing this out to us, in a private communication.) We do not know whether all "cofinal embedders" are minimal. We can show, however, that proper cofinal extension to a Nerode semiring is both possible and somewhat limited, for 2-correct structures.

The present section is centered around the proof of Lemma 3.3 below; that lemma is a four-quantifier analogue (albeit not a perfect four-quantifier analogue) of the main result of [G], and our proof of it consists in constructing a suitable modification of Gaifman's proof. From here on, we shall use the term "n-correct" of Definition 2.1 in a more general way, dropping the requirement on $\mathcal{M}$ that it be a model of *PA*.

To begin with, it is possible to have a nontrivial cofinal embedding of one Nerode semiring in another. The following routine lemma is a preliminary to establishing this fact:

3.1. Lemma. *Let $\bar{a}$ and $\bar{c}$ be "new constants"; let $\{\phi_i \mid i \in \omega\}$ be a listing of the unary recursive functions; and let $\Gamma = \{\psi \mid \psi$ is an ω-true $\forall\exists$ sentence of the base language$\} \cup \{\bar{0} < \bar{a} < \bar{c}, \bar{1} < \bar{a}, \dots\} \cup \{\phi_i(\bar{c}) \neq \bar{a} \mid i \in \omega\}$. Then Γ is finitely satisfiable in ω, and hence has a countable model.*

Proof. Let Δ be any finite subset of Γ; let $\bar{n}_0$ be the largest "ordinary numeral" appearing in any element of Δ; and let k be the largest number such that ϕ_k appears in an element of Δ. (Choose $\bar{n}_0$, respectively k, arbitrarily if there are no such appearances.) Interpret $\bar{c}$, in ω, by the number $n_0 + k + 3$. Then there exists $u \in \omega$ such that $n_0 + 1 \leqq u < n_0 + k + 3$ & $u \notin$

$\{\phi_i(n_0 + k + 3) \mid i \leqq k\}$. Interpreting $\bar{a}$ by such a u, we have the satisfaction of Δ in ω.

3.2. THEOREM. *There exist Nerode semirings* N(X) *and* N(Y) *such that* N(X) *is a proper cofinal substructure of* N(Y).

PROOF. Let Γ be as in the preceding lemma, and let $\mathcal{M}$ be a countable model of Γ. Consider the one-generator submodels $\mathcal{M}(c)$ and $\mathcal{M}(2^a 3^c)$ generated (via the $\mathcal{M}$-recursive functions) within Γ via the interpretations c and a of $\bar{c}$ and $\bar{a}$, respectively. Since $\mathcal{M}(2^a 3^c) \cong \mathrm{N}(Y)$ for an appropriately chosen isol Y, with $2^a 3^c$ corresponding to Y, it suffices to verify that $\mathcal{M}(c)$ is a proper cofinal submodel of $\mathcal{M}(2^a 3^c)$. Clearly, $\mathcal{M}(c) \subseteq \mathcal{M}(2^a 3^c)$. Moreover, since Lemma 3.1 implies that $a \neq \phi(c)$ for each $\mathcal{M}(2^a 3^c)$-recursive function ϕ, we have $\mathcal{M}(c) =$ *a proper* substructure of $\mathcal{M}(2^a 3^c)$. (Be it noted, in this connection, that the $\mathcal{M}(2^a 3^c)$-recursive functions are just the restrictions to $\mathcal{M}(2^a 3^c)$ of the $\mathcal{M}$-recursive functions, which in turn are just the liftings to $\mathcal{M}$ of the ordinary recursive functions.) Now, clearly, $2^a 3^c < 2^c 3^c \in \mathcal{M}(c)$. Additionally, it is clear that if f is any *nondecreasing* recursive function and $x \leqq y$ holds in $\mathcal{M}(2^a 3^c)$, then $f(x) \leqq f(y)$ holds in $\mathcal{M}(2^a 3^c)$; for, $\forall x \forall y[x \leqq y \rightarrow f(x) \leqq f(y)]$ can clearly be written as an ω-true $\forall\exists$ statement. In particular, if $f\colon \omega \rightarrow \omega$ is *any* given recursive function, and if we define $g_f(x) = \Sigma_{y \leqq x} f(y)$, then, in $\mathcal{M}(2^a 3^c)$, we have:

$$f(2^a 3^c) \leqq g_f(2^a 3^c) \leqq g_f(2^c 3^c) \in \mathcal{M}(c).$$

(We here apply also, of course, the fact that since ω satisfies $\forall x(f(x) \leqq g_f(x))$, so does $\mathcal{M}(2^a 3^c)$.) Since $f(2^a 3^c)$ is an arbitrary element of $\mathcal{M}(2^a 3^c)$, it follows that $\mathcal{M}(c)$ is cofinal in $\mathcal{M}(2^a 3^c)$, and we are done.

Now for our main consideration.

Recall that if $\mathcal{N}$, $\mathcal{M}$ are $\mathcal{L}$-structures with $\mathcal{N} \subseteq \mathcal{M}$, then $\mathcal{N}$ is said to be an *n-elementary substructure* of $\mathcal{M}$ if for every prenex normalform $\mathcal{L}$-formula $\phi(x_1, \dots, x_k)$ having at most n alternating blocks of quantifiers, we have $\boldsymbol{a} \in |\mathcal{N}|^k \Rightarrow (\mathcal{N} \models \phi(\boldsymbol{a}) \Leftrightarrow \mathcal{M} \models \phi(\boldsymbol{a}))$. (This definition can, of course, be extended to refer to $\mathcal{L}'$-structures and formulae.)

3.3. LEMMA. *Let $\mathcal{N}$ and $\mathcal{M}$ be 2-correct $\mathcal{L}$-structures such that $\mathcal{N}$ is a cofinal substructure of $\mathcal{M}$. Suppose further that $\mathcal{N} \models PA_4$. Then $\mathcal{N}$ is a 3-elementary substructure of $\mathcal{M}$.*

PROOF. Throughout our proof, we shall be viewing $\mathcal{N}$ and $\mathcal{M}$ as $\mathcal{L}'$-structures, via π_D. This will allow us to "contract quantifiers" in both $\mathcal{N}$ and

$\mathcal{M}$, via the respective extensions of $z = (x)_y$ to $\mathcal{N}$ and to $\mathcal{M}$. In particular, if $k \geqq 1$ then each of $\mathcal{N}$, $\mathcal{M}$, so viewed, satisfies the sentence $\forall x_1 \cdots \forall x_k \exists y[(y)_1 = x_1 \wedge \cdots \wedge (y)_k = x_k]$. It follows, for example, that if ψ is the $\forall\exists\forall$ prenex normalform $\mathcal{L}$-formula

$$\forall x_1 \cdots \forall x_j \exists y_1 \cdots \exists y_l \forall z_1 \cdots \forall z_t \phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}, w_1, \ldots, w_r),$$

then $\psi \leftrightarrow \psi'$ is valid in both $\mathcal{N}$ and $\mathcal{M}$, where ψ' is the formula

$$\forall x \exists y \forall z \phi((x)_1, \ldots, (x)_j, (y)_1, \ldots, (y)_l, (z)_1, \ldots, (z)_t, \boldsymbol{w}).$$

This implies that if $a_1, \ldots, a_r \in |\mathcal{N}|$ and $\mathcal{N} \models \psi(\boldsymbol{a})$, then $\mathcal{N} \models \psi'(\boldsymbol{a})$; moreover, $\mathcal{M} \models \psi'(\boldsymbol{a}) \Rightarrow \mathcal{M} \models \psi(\mathbf{a})$. So, if it could be shown that $\mathcal{N} \models \psi'(\boldsymbol{a}) \Rightarrow \mathcal{M} \models \psi'(\boldsymbol{a})$, we could conclude the 3-elementary character of $\mathcal{N}$ as a substructure of $\mathcal{M}$; such, therefore, will be the strategy of the proof.

With these observations in mind, we shall now proceed by way of suitable minor adjustments to portions of Gaifman's proof of [G, Th. 3]. The reader should have [G] in hand, since we shall often merely reference a section of the proof of [G, Th. 3], rather than reproduce that section.

As noted by Gaifman on p. 135 of [G], in his proof of the implication (3)⇒(4) (see [G, p. 134] for the statements of conditions (1) through (4)), the sentence

$$\tau = \forall \boldsymbol{x}[\forall \boldsymbol{u} \leqq \boldsymbol{u}' \exists \boldsymbol{v} \phi \rightarrow \exists \boldsymbol{v}' \forall \boldsymbol{u} \leqq \boldsymbol{u}' \exists \boldsymbol{v} \leqq \boldsymbol{v}' \phi]$$

is provable in PA (it is, of course, just an instance of the Collection Schema); here, for our present purposes, ϕ is to be a formula *of* $\mathcal{L}$. If ϕ is quantifier-free, then τ is logically equivalent to a sentence in PA_4 (indeed, in PA_3), and hence $\mathcal{N} \models \tau$. Gaifman's proof of (3)⇒(4) therefore works, in our present context, *as written*, when we replace his pair M_1, M_2 by the pair $\mathcal{N}$, $\mathcal{M}$. In fact, however, Gaifman's condition (3) *holds for the pair* $\mathcal{N}$, $\mathcal{M}$; this is because, by *DPRM*, *bounded* $\mathcal{L}$-formulae are absolute for 2-correct $\mathcal{L}$-structures. Hence, Gaifman's condition (4) also holds for $\mathcal{N}$, $\mathcal{M}$; i.e., if $\forall \boldsymbol{x} \exists \boldsymbol{y} \phi(w_1, \ldots, w_k, \boldsymbol{x}, \boldsymbol{y})$ is an arbitrarily given $\forall\exists$ prenex formula of $\mathcal{L}$ then, for each $\boldsymbol{a} \in |\mathcal{N}|^k$, we have that

$$\mathcal{N} \models \forall \boldsymbol{x} \exists \boldsymbol{y} \phi(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}) \Rightarrow \mathcal{M} \models \forall \boldsymbol{x} \exists \boldsymbol{y} \phi(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}).$$

It follows that $\mathcal{N}$ is, at any rate, a *2*-elementary substructure of $\mathcal{M}$, relative to $\mathcal{L}$-formulae. The remainder of the proof consists in showing that this extends to *all* $\forall\exists\forall$ *prenex formulas of* $\mathcal{L}'$. It turns out that we can achieve this extension by employing Gaifman's proof of [G, Theorem 3, implication (4)$\Rightarrow$(1)] *almost verbatim*.

Let τ, then, be an $\forall\exists\forall$ prenex formula of $\mathscr{L}'$, with free variables $x_1, \ldots, x_k$. As noted at the outset, we may assume τ to be of the form

$$\forall u \exists v \forall w \phi(\boldsymbol{x}, u, v, w).$$

Let $\boldsymbol{a} \in |\mathscr{N}|^k$; and suppose that $\mathscr{N} \models \forall u \exists v \forall w \phi(\boldsymbol{a}, u, v, w)$. Let b be any element of $|\mathscr{M}|$. Applying the cofinality of $\mathscr{N}$ in $\mathscr{M}$, let $b_0 \in |\mathscr{N}|$ be such that $b \leqq b_0$. Clearly, then, we have $\mathscr{N} \models \forall u \leqq b_0 \exists v \forall w \phi(\boldsymbol{a}, u, v, w)$. As noted in [G],

$$PA^{\mathscr{L}'_0} \vdash \forall y \forall x [\forall u \leqq x \exists v \forall w \phi(y, u, v, w) \rightarrow \exists z \forall u \leq x \forall w \phi(y, u, (z)_u, w)];$$

moreover, as a moment's manipulation shows, this latter sentence is logically equivalent to a Π^0_4 sentence of $\mathscr{L}'$ and hence, by Proposition 1.2, holds in $\mathscr{N}$. Thus, $\mathscr{N} \models \exists z \forall u \leqq b_0 \forall w \phi(\boldsymbol{a}, u, (z)_u, w)$. Let $c \in |\mathscr{N}|$ be such that $\mathscr{N} \models \forall u \leqq b_0 \forall w \phi(\boldsymbol{a}, u, (c)_u, w)$.

Now (once again applying the translation algorithm of [G, p. 137] and the fact that both $\mathscr{N}$ and $\mathscr{M}$ are 2-correct), there is an existential (i.e., $\exists$ prenex) formula $\zeta = \exists y \psi(\boldsymbol{x}, u, v, w, \boldsymbol{y})$ *of* $\mathscr{L}$ such that

$$\forall \boldsymbol{x} \forall u \forall v \forall w [\phi(\boldsymbol{x}, u, (v)_u, w) \leftrightarrow \zeta(\boldsymbol{x}, u, v, w)]$$

holds in both $\mathscr{N}$ and $\mathscr{M}$. Thus, $\mathscr{N} \models \forall u \leqq b_0 \forall w \exists \boldsymbol{y} \psi(\boldsymbol{a}, u, c, w, \boldsymbol{y})$. So, by what was shown earlier, also $\mathscr{M} \models \forall u \leqq b_0 \forall w \exists \boldsymbol{y} \psi(\boldsymbol{a}, u, c, w, \boldsymbol{y})$. Hence $\mathscr{M} \models \forall u \leqq b_0 \forall w \phi(\boldsymbol{a}, u, (c)_u, w)$. But $b \leqq b_0$; so $\mathscr{M} \models \forall w \phi(\boldsymbol{a}, b, (c)_b, w)$. Since b was chosen *arbitrarily* from $|\mathscr{M}|$, we therefore have

$$\mathscr{M} \models \forall u \exists v \forall w \phi(\boldsymbol{a}, u, v, w),$$

and the proof of the lemma is complete.

3.4. THEOREM. *Let* N(X) *be a Nerode semiring. Then no nonstandard* 2-*correct model* $\mathscr{N}$ *of* PA_4 *can be cofinally embedded in* N(X).

PROOF. By Lemma 3.3, if such an $\mathscr{N}$ were so embedded, it would be a 3-elementary substructure of N(X). It would now suffice to simply cite [F–S–T], as proved in [L]; however, we prefer to proceed directly (providing, as we do so, what amounts to a proof of [F–S–T]). In his proof of [H, Th. 2.6], Hirschfeld showed that a certain Π^0_3 $\mathscr{L}$-predicate $\phi(z)$ defines ω in every recursive ultrapower. Hence, $\forall z \phi(z)$ is a Π^0_3 sentence true in ω but (since N(X) is isomorphic to a recursive ultrapower) false in N(X). If we can select $\phi(z)$ in such a way that, additionally, $PA \vdash \forall z \phi(z)$, then we can contradict the assertion that $\mathcal{N}$ is a 3-elementary substructure of $\mathbf{N}(X)$; for, it follows from *DPRM* that there is an $\forall \exists \forall$ prenex normalform sentence τ such that $\tau \leftrightarrow \forall z \phi(z)$ holds in every 2-correct $\mathcal{L}$-structure. But in fact $\phi(z)$ *can* be so chosen: if $T(i, x, y)$ and $U(x)$ are the standard T-predicate and "output function" as defined, say, in [Da], then $T(i, x, y)$ and $U(x)$ are primitive recursive. Hence $U(x)$ is provably recursive, as is the characteristic function of $T(i, x, y)$. Moreover, we can prove in *PA* that $\forall i \forall x \forall y \forall z[(T(i, x, y) \wedge T(i, x, z) \rightarrow y = z]$; and we can derive in *PA* the existence of functions $z(i, x)$ and $y(z, x)$ whose infomal descriptions are as follows:

$$z(i, x) = \begin{cases} U(\min yT(i, x, y)) + 1, & \text{if } \exists yT(i, x, y), \\ 1, & \text{otherwise;} \end{cases}$$

$$y(z, x) = \Pi_{i=0}^{z} z(i, x).$$

Having so defined $y(z, x)$, we can prove in *PA* that, for every z and every x, $\forall i \forall w[(i \leqq z \wedge \exists u(T(i, x, u) \wedge w = U(u))) \rightarrow w < y(z, x)]$. If we then define $\phi(z)$ as $\forall x \exists y \forall i \forall w[(i \leqq z \wedge \exists u(T(i, x, u) \wedge w = U(u))) \rightarrow w < y]$, we have that $PA \vdash \forall z \phi(z)$, and hence that $\mathcal{N} \models \forall z \phi(z)$; moreover, as is easily verified using Hirschfeld's line of argument, $\phi(z)$ defines ω in every recursive ultrapower. Theorem 3.4 is therefore proved.

(For a closely related approach to selecting an appropriate $\phi(z)$, see Lerman's proof ([L, Th. 2.1]) of the Feferman–Scott–Tennenbaum theorem [F–S–T]. Given [H] as a starting point, our path to a suitable sentence is shorter than Lerman's: the lengthy technical lemma [L, Lemma 2.1] is not needed, and no familiarity with [L]'s treatment of recursive ultrapowers in terms of indecomposability is required. Moreover, it can be shown that Lerman's sentence is *not* the universal closure of a formula that defines ω in each recursive ultrapower. The perspective of [L], on the other hand, has its own considerable merits; the reader not already familiar with it should certainly take a look.)

We remark that in Lemma 3.3 and Theorem 3.4 we can replace "2-correct $\mathcal{L}$-structure" by "$\forall \exists$-correct $\mathcal{L}$-structure"; i.e., we can without loss of generality speak simply of those $\mathcal{L}$-structures, the "$\forall \exists$-correct" ones, that satisfy all $\forall \exists$ prenex normalform $\mathcal{L}$-sentences τ such that $\omega \models \tau$. This is because (see [H, Cor. 1.7.1]) ω-true Π_2^0 arithmetic is an inductive theory and hence equivalent to a theory all of whose axioms are $\forall \exists$ prenex normalform sentences.

Also, so long as we are working within the class of 2-correct $\mathcal{L}$-structures, *DPRM* allows us to enlarge our notion of "k-elementary," for substructures, to cover Π_k^0 $\mathscr{L}$-sentences in general (rather than just the k-quantifier $\forall\exists\cdots$ *prenex* sentences).

4. Unions and intersections of $\forall\exists$-correct semirings of isols

In view of [H, Th. 1.8], [McL$_3$, Lemmas 2 and 3], and the Hirschfeld–Nerode Representation Theorem ([H, Th. 3.2]), we see that every countable 2-correct $\mathscr{L}$-structure $\mathscr{M}$ is isomorphic to the union of a chain $\mathrm{N}(X_0) \subseteq \mathrm{N}(X_1) \subseteq \mathrm{N}(X_2) \subseteq \cdots$ of Nerode semirings. Furthermore, by [H–W, Ch. 1, Lemma 1.8], we have (since ω is existentially complete for the class of 2-correct $\mathscr{L}$-structures, and is an initial segment of any such structure) that the 2-correct $\mathscr{L}$-structures enjoy "joint embedding"; i.e., if $\mathscr{M}_1$ and $\mathscr{M}_2$ are any two 2-correct $\mathscr{L}$-structures then there is a 2-correct $\mathscr{L}$-structure $\mathscr{M}_3$ such that each of $\mathscr{M}_1, \mathscr{M}_2$ is isomorphic to a submodel of $\mathscr{M}_3$. Since the ability to "glue together" pairs of Nerode semirings, more-or-less at will, would be a very useful technical tool, it is natural to wonder whether the union of two $\forall\exists$-correct semirings of isols is always *extendible to* some $\forall\exists$-correct semiring of isols, thus giving (*within* Λ) an optimal form of joint embedding. Such, unfortunately, is not the case.

To see this, we begin by noting a sharper version of [McL$_3$, Th. 6(ii)]:

4.1. THEOREM. (i) *No uncountable subset of the set Λ_R of all regressive isols is a chain (with respect to isolic $\leqq$).*

(b) *No uncountable subset of Λ_R is a semiring.*

PROOF. Part (a), like [McL$_3$, Th. 6(ii)], is an immediate consequence of an embedding theorem due to Ellentuck ([McL$_1$, Th. 16.13]) which implies the countability of all chains of regressive isols. It is not necessary, however, to appeal to anything so exotic as Ellentuck's embedding theorem: both (a) and (b) can be dealt with very straightforwardly in terms of Dekker's theory ([De]) concerning the *degrees* of regressive isols.

Recall the definition of the *degree*, Δ_X, of a regressive isol X: $\Delta_X =$ the (Turing) degree of any *retraceable* element of X. (Each two retraceable sets $\alpha, \beta \in X$ are of the same degree $\mathbf{d}$, and $\mathbf{d}$ is *minimal* among the degrees of sets $\gamma \in X$.) Then (see [De]) we have: (1) $X \leqq Y \Rightarrow \Delta_X = \Delta_Y$, for any two *infinite* regressive isols X and Y; and (2) $X + Y \in \Lambda_R \Rightarrow \Delta_X = \Delta_Y$, for any two *infinite* regressive isols X and Y.

To prove (a), suppose $\{X_\alpha \mid \alpha \in A\}$ is a chain of regressive isols, with respect to the $\leqq$ relation among isols. If $\{X_\alpha \mid \alpha \in A\}$ is uncountable, then, since each degree contains only $\aleph_0$ sets and distinct isols are disjoint, there must be *infinite* elements X_β, X_γ of the chain such that $\Delta_{X_\beta} \neq \Delta_{X_\gamma}$. But, by (1), this is impossible. As for (b), if $\mathbf{F} = \langle \{X_\alpha \mid \alpha \in A\}, +, \cdot \rangle$ were an uncountable semiring of regressive isols, then once again there would of necessity be *infinite* isols $X_\beta, X_\gamma \in \{X_\alpha \mid \alpha \in A\}$ such that $\Delta_{X_\beta} \neq \Delta_{X_\gamma}$. (2), however,then yields $X_\beta + X_\gamma \notin \{X_\alpha \mid \alpha \in A\}$, contradicting the assumption that **S** is a semiring.

Theorem 4.1(b) suggests the possibility that the full collection $\Lambda(\forall\exists) = \bigcup_{X \in RST} \mathrm{N}(X) = \{X \mid X \text{ is finite or RST}\}$ discussed in [McL$_3$] is not a semiring, and may not even be closed under addition. This is, indeed, the case. Before proving it, we observe the following corollary to 4.1:

4.2. Corollary. *There exist Nerode semirings* N(X) *and* N(Y), *with X and Y regressive, such that* $\mathrm{N}(X) \cup \mathrm{N}(Y)$ *is not contained in any* $\forall\exists$*-correct semiring of isols.*

Proof. Suppose $\mathrm{N}(X) \cup \mathrm{N}(Y) \subseteq \mathcal{M}$, where X, Y are RST isols and $\mathcal{M}$ is an $\forall\exists$-correct subsemiring of Λ. Then, since $\mathcal{M}$ is linearly ordered by isolic $\leqq$, we have either $X \leqq Y$ or $Y \leqq X$. But (by [N$_3$, proof of Th. 4.4]) there are $2^{\aleph_0}$ regressive isols that are RST. Therefore, by Theorem 4.1, there must be some pair X, Y of RST regressive isols that are not $\leqq$-comparable. For such a pair, then, there can be no $\forall\exists$-correct semiring $\mathcal{M} \subseteq \Lambda$ such that $\mathrm{N}(X) \cup \mathrm{N}(Y) \subseteq \mathcal{M}$.

4.3. Theorem. $\Lambda(\forall\exists)$ *is not a semiring. Indeed, there are infinite* regressive *isols X and Y in $\Lambda(\forall\exists)$ such that neither XY nor $X+Y$ is RST.*

Proof. First note that we cannot merely apply a degree argument as in the proof of Theorem 4.1, since that is limited to killing regressiveness and does not address nonmembership in $\Lambda(\forall\exists)$. Now, applying Corollary 4.2, let A and B be regressive RST isols such that $\mathrm{N}(A) \cup \mathrm{N}(B)$ has no extension to an $\forall\exists$-correct semiring of isols. We claim that $X = 2^{2^{2A}}$ and $Y = 2^{2^{3^{2B}}}$ satisfy the requirements of the theorem. That X and Y, so defined, are regressive and RST follows from the obvious relations $2^{2^{2A}} \in \mathrm{N}(A)$, $2^{2^{3^{2B}}} \in \mathrm{N}(B)$. (It is a consequence of Barback's theorem [McL$_1$, Lemma 18.24], applied to the "combinatorial constituents" of a recursive function, that if $Z \in \Lambda_R$, Z RST, then $\mathrm{N}(Z) \subseteq \Lambda_R$.) Recall that if $f: \omega \to \omega$ is any recursive function, then $f = f^+ - f^-$, where f^+ and f^- are two naturally defined *recursive combinatorial functions*. Now, there are recursive functions $g: \omega \to \omega$, $h: \omega \to \omega$, $r: \omega \to \omega$, and $s: \omega \to \omega$ such that the following are true:

$$\omega \models \forall x \forall y \forall z[z = 2^{2x} + 2^{3^{2y}} \rightarrow x + g^-(z) = g^+(z)];$$

$$\omega \models \forall x \forall y \forall z[z = 2^{2x} + 2^{3^{2y}} \rightarrow y + h^-(z) = h^+(z)];$$

$$\omega \models \forall x \forall y \forall z[z = 2^{2^{2x}} + 2^{2^{3^{2y}}} \rightarrow x + r^-(x) = r^+(z)];$$

$$\omega \models \forall x \forall y \forall z[z = 2^{2^{2x}} + 2^{2^{3^{2y}}} \rightarrow y + s^-(z) = s^+(z)].$$

It follows, from the Basic Nerode Metatheorem ([N_1]; see also [McL_1, Ch. 12]), that if $Z = X + Y = 2^{2^{2A}} + 2^{2^{3^{2B}}}$ then $A = r_\Lambda(Z)$ and $B = s_\Lambda(Z)$, so that Z $RST \Rightarrow N(A) \cup N(B) \subseteq N(Z)$, an impossibility. Thus, $X + Y \notin \Lambda(\forall\exists)$. Suppose $W = XY = 2^{2^{2A}} \cdot 2^{2^{3^{2B}}}$ were in $\Lambda(\forall\exists)$. Since $2^{2^{2A}} \cdot 2^{2^{3^{2B}}} = 2^{(2^{2A} + 2^{3^{2B}})}$, this would clearly imply $2^{2A} + 2^{3^{2B}} \in N(W) \subseteq \Lambda(\forall\exists)$. But then, arguing as above with g and h in place of r and s, we obtain $\{A, B\} \subseteq N(W)$, and a contradiction once again ensues. So, $XY \notin \Lambda(\forall\exists)$.

In contrast to Corollary 4.2, we have:

4.4. Proposition. *The intersection of any two $\forall\exists$-correct semirings of isols is an $\forall\exists$-correct semiring.*

Proof. Let $\mathbf{R}_1 \subseteq \Lambda$ and $\mathbf{R}_2 \subseteq \Lambda$ be two $\forall\exists$-correct semirings. Clearly, $\mathbf{R}_1 \cap \mathbf{R}_2$ is again a semiring (containing ω). To verify $\forall\exists$-correctness, we need only show ([H, Th. 1.8]) that $\mathbf{R}_1 \cap \mathbf{R}_2$ is closed under all $\mathbf{R}_1$-recursive functions. By [McL_3, Lemma 2], however, this is equivalent to showing that $\mathbf{R}_1 \cap \mathbf{R}_2$ is closed under $f_\Lambda \lceil_{\mathbf{R}_1^k \cap \mathbf{R}_2^k}$ for every recursive function $f: \omega^k \rightarrow \omega$, $k \geqq 1$. $\mathbf{R}_1$, however, is closed under $f_\Lambda \lceil_{\mathbf{R}_1^k}$ for each such f ([McL_3, Th. 2]); and, likewise, $\mathbf{R}_2$ is closed under $f_\Lambda \lceil_{\mathbf{R}_2^k}$ for each such f. It follows that $\mathbf{R}_1 \cap \mathbf{R}_2$ has the required closure property.

Remark. Proposition 4.4 is closely related to [H, Cor. 1.8.2]. Note, however, that we are not (in view of Corollary 4.2) assuming $\mathbf{R}_1 \cup \mathbf{R}_2$ to be embedded in a 2-correct simultaneous extension model.

Having noted that $N(X) \cap N(Y)$ behaves well, it is natural to ask whether X and Y can be chosen so that $N(X) \cap N(Y)$ is as small as possible, i.e., $= \omega$. The answer, not very surprisingly, is *yes*, even when it is required that X and Y be regressive. This follows from [H, §4.4], with the aid of [McL_2, Th. 1.1], [N_3, §4], and Proposition 4.4. We shall, however, proceed by way of a more general result on intersections.

4.5. Theorem. *Let $\mathcal{M}$ be any countable 2-correct model of PA. Then there* *exist countable*, $\forall\exists$*-correct semirings* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *of regressive isols such that* $\mathbf{R}_1 \not\subseteq \mathbf{R}_2$, $\mathbf{R}_2 \not\subseteq \mathbf{R}_1$, *and* $\mathbf{R}_1 \cap \mathbf{R}_2 \cong \mathscr{M}$.

PROOF. The proof is a Diagram Lemma argument combined with results from [McL$_2$] and [N$_3$]. To begin, recalling the MacDowell–Specker theorem, let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two countable, proper, elementary end extensions of $\mathscr{M}$, such that $|\mathscr{M}_1| \cap |\mathscr{M}_2| = |\mathscr{M}|$. Adjoin new constants $a_0, a_1, a_2, \ldots$ and $c_0, c_1, c_2, \ldots$ to $\mathscr{L}$. In the so-expanded language, write down the diagram, diag($\mathscr{M}_1$), of $\mathscr{M}_1$, using $a_0, a_1, a_2, \ldots$ to denote the elements of $|\mathscr{M}| - \omega$ and $c_0, c_2, c_4, \ldots$ to denote the elements of $|\mathscr{M}_1| - |\mathscr{M}|$. Similarly, write down diag($\mathscr{M}_2$), denoting the elements of $|\mathscr{M}_2| - |\mathscr{M}|$ by $c_1, c_3, c_5, \ldots$ and the elements of $|\mathscr{M}| - \omega$ in exactly the same way as in the case of diag($\mathscr{M}_1$). Let Δ_1 = the set of all ω-true Π^0_2 sentences of $\mathscr{L}$, $\Delta_2 = \{c_{2k} \neq c_{2j+1} \mid k, j \in \omega\}$, and $\Gamma = \Delta_1 \cup \Delta_2 \cup \text{diag}(\mathscr{M}_1) \cup \text{diag}(\mathscr{M}_2)$. Now, every finite subset of Γ is realizable in $\mathscr{M}$. (This is not hard to show, but requires a little care: we need to observe the facts that (a) the c_i's denote only elements that are bigger than all elements of $\mathscr{M}$, and (b) $\mathscr{M}_i$ is an *elementary* end-extension of $\mathscr{M}$, $i = 1, 2$; then, with (a) and (b) in mind, we need to work first on $\mathscr{M}_1$, and then on $\mathscr{M}_2$ (or vice-versa).) Thus, Γ has a countable model $\mathscr{M}^*$. $\mathscr{M}^*$ is 2-correct, and (isomorphic copies of) $\mathscr{M}_1$ and $\mathscr{M}_2$ sit inside $\mathscr{M}^*$ with $\mathscr{M}$ as their common part. Now ([McL$_2$, Th. 1.1]) embed $\mathscr{M}^*$ in a Nerode semiring N(X). By results in [N$_3$], N(X) is isomorphic to a Nerode semiring with a regressive generator; so we will go ahead and assume that X is regressive (and hence that all elements of N(X) are regressive). Since $\mathscr{M}_i$ is 2-correct, $i = 1, 2$, $\mathscr{M}_1$ and $\mathscr{M}_2$ appear in N(X) as $\forall\exists$-correct subsemirings $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. Clearly, $\mathbf{R}_1$ and $\mathbf{R}_2$ have the properties cited in the statement of the theorem.

4.6. COROLLARY. (i) *There exist Nerode semirings* N(X), N(Y), *with* $X, Y \in \Lambda_R$, *such that* $\text{N}(X) \cap \text{N}(Y) = \omega$.

(ii) *There exist Nerode semirings* N(X), N(Y), *with* $X, Y \in \Lambda_R$, *such that* $\text{N}(X) \not\subseteq \text{N}(Y)$, $\text{N}(Y) \not\subseteq \text{N}(X)$, *and* $\text{N}(X) \cap \text{N}(Y) \neq \omega$.

PROOF. For (i), let $\mathscr{M} = \omega$ in Theorem 4.5. Let $X \in \mathbf{R}_1 - \omega$, $Y \in \mathbf{R}_2 - \omega$. By [H, Th. 1.8] and [McL$_3$, Lemma 2], $\text{N}(X) \subseteq \mathbf{R}_1$ and $\text{N}(Y) \subseteq \mathbf{R}_2$. Since $\mathbf{R}_1 \cap \mathbf{R}_2 = \omega$, $\text{N}(X) \cap \text{N}(Y) = \omega$ also. For (ii), let $\mathscr{M}$, in 4.5, be chosen nonstandard. Let $X \in \mathbf{R}_1 - (\mathbf{R}_1 \cap \mathbf{R}_2)$, $Y \in \mathbf{R}_2 - (\mathbf{R}_1 \cap \mathbf{R}_2)$, $Z \in \mathbf{R}_1 \cap \mathbf{R}_2$. Then (by [H, Th. 1.8], [McL$_3$, Lemma 2], and Proposition 4.4), we have $2^X 3^Z \in \mathbf{R}_1 - (\mathbf{R}_1 \cap \mathbf{R}_2)$ and $2^Y 3^Z \in \mathbf{R}_2 - (\mathbf{R}_1 \cap \mathbf{R}_2)$. So, since $Z \in \text{N}(2^X 3^Z) \cap \text{N}(2^Y 3^Z)$, we have (again using [H, Th. 1.8] and [McL$_3$, Lemma 2]) that the semirings $N(2^X 3^Z)$ and $N(2^Y 3^Z)$ are as required.

REMARK. We may of course take $\mathbf{R}_1 \cong \mathbf{R}_2$ in Theorem 4.5, and this in turn enables us to arrange that $N(X) \cong N(Y)$ in Corollary 4.6(i).

To conclude this section, we shall record a theorem that we think is interesting, although it has a certain problematic aspect. The interest lies in its showing that the regressive elements of an $\forall\exists$-correct semiring $\mathcal{M}$ of isols must sit very nicely in $\mathcal{M}$; the problem is that we do not know at present (see §6) whether the theorem has any nontrivial instances.

4.7. THEOREM. *Let $\mathcal{M}$ be an $\forall\exists$-correct semiring of isols; and let $\mathcal{M}_R$ be the set of all regressive elements of $\mathcal{M}$. Then:*

(1) $\mathcal{M}_R$ is a countable initial segment of $\mathcal{M}$, and

(2) $\mathcal{M}_R$ is itself a model of $\forall\exists$ arithmetic.

PROOF. Since Λ_R is closed under isolic predecessor, (1) follows from Theorem 4.1(a). In view of [H, Th. 1.8] and [McL$_3$, Lemma 2], it will suffice for a proof of (2) to verify that $\mathcal{M}_R$ is closed under $f_\Lambda \lceil \mathcal{M}_R^k$, for every recursive function $f: \omega^k \to \omega$, $k \geqq 1$. So, suppose that $f: \omega^k \to \omega$ is recursive. Let a recursive function $g: \omega \to \omega$ be defined thus:

$$g(x) = \sum_{x_1 \leqq x, \ldots, x_k \leqq x} f(x_1, \ldots, x_k).$$

Now, let $X_1, \ldots, X_k \in \mathcal{M}_R$. Then, applying [McL$_3$, Th. 2] and Nerode's General Composition Theorem ([McL$_1$, Th. 19.6]), we see that the following two statements hold in $\mathcal{M}$:

(a) $\max_\Lambda(X_1, \ldots, X_k) = X_1 \vee \cdots \vee \max_\Lambda(X_1, \ldots, X_k) = X_k$;

(b) $f_\Lambda(X_1, \ldots, X_k) \leqq g_\Lambda(\max_\Lambda(X_1, \ldots, X_k)) = (g \circ \max)_\Lambda(X_1, \ldots, X_k)$.

By (a), we have

$$\max_\Lambda (X_1, \ldots, X_k) \in \mathcal{M}_R.$$

Hence also $g_\Lambda(\max_\Lambda(X_1, \ldots, X_k)) \in \mathcal{M}_R$, since $\mathcal{M}$ is closed under the action of the $\mathcal{M}$-recursive function g_Λ and since (again using Barback's theorem [McL$_1$, Lemma 18.24] cited earlier) the image of a *regressive* isol Y under the *unary* function g_Λ *is regressive* given that $g_\Lambda(Y)$ is defined at all. Thus, by (b) and the fact that Λ_R is closed under isolic predecessor, we have $f_\Lambda(X_1, \ldots, X_k) \in \mathcal{M}_R$, as required. ($f_\Lambda(X_1, \ldots, X_k) \in \mathcal{M}$, since $\mathcal{M}$ is $\forall\exists$-correct.)

5. Extendibility of Nerode semirings

As a consequence of the results in [N_3, §5], specifically [N_3, Th. 5.1], we know that every Nerode semiring can be embedded in a countable correct model (i.e., a countable model of the set of all ω-true $\mathscr{L}$-sentences). The question then naturally arises (cf. [McL_2]): can at least some of the Nerode semirings be embedded as *initial segments* of countable correct models or even, merely, of models of *PA*? The answer is "no."

With the help of ideas from [H] and [H–W, Ch. 8], we shall prove a theorem that establishes this negative result in a rather sharp form relative to the fragments PA_n. The following technical lemma is of some assistance:

5.1. LEMMA. *Let $p \subseteq \omega^{n+1}$, $n \geqq 1$, be a partial recursive function. Then there exists a Σ_1^0 $\mathscr{L}$-formula $\phi(\boldsymbol{x}, y)$, where $\boldsymbol{x}$ = the sequence $x_1, \ldots, x_n$, such that*

(i) *$\phi(\boldsymbol{x}, y)$ defines p in ω, and*

(ii) *$PA \vdash \forall \boldsymbol{x} \forall y \forall u[(\phi(\boldsymbol{x}, y) \wedge \phi(\boldsymbol{x}, u)) \rightarrow y = u]$.*

(*Thus, p is "provably univalent, via ϕ."*)

PROOF. Suppose p is defined, in ω, by $\exists z_1 \cdots \exists z_k B(\boldsymbol{x}, y, z_1, \ldots, z_k)$, B a *bounded* $\mathscr{L}$-formula. (See, e.g., [P, "Fact 8"].) Let a formula $B^*(\boldsymbol{x}, y, z)$ be defined thus:

$$B^*(\boldsymbol{x}, y, z) \leftrightarrow y \leqq z \wedge \exists w_1 \leqq z \cdots \exists w_k \leqq z B(\boldsymbol{x}, y, w_1, \ldots, w_k).$$

Next, let $B_0(\boldsymbol{x}, y, z)$ be defined by

$$B_0(\boldsymbol{x}, y, z) \leftrightarrow B^*(\boldsymbol{x}, y, z) \wedge \forall v \leqq z \forall t \leqq v[t \neq y \rightarrow \neg B^*(\boldsymbol{x}, t, v)].$$

Finally, let $\phi(\boldsymbol{x}, y)$ be the formula $\exists z B_0(\boldsymbol{x}, y, z)$. Then it is easily verified that ϕ defines p in ω; and it is, moreover, clear that the resources of *PA* suffice to establish the statement

$$\forall \boldsymbol{x} \forall y \forall u[(\phi(\boldsymbol{x}, y) \wedge \phi(\boldsymbol{x}, u)) \rightarrow y = u].$$

5.2. THEOREM. *No Nerode semiring admits end-extension to a model of PA_3.*

PROOF. Suppose $N(X) \subset_e \mathscr{M}$ holds, where X is an *RST* isol and $\mathscr{M} \models PA_3$. (As is the standard practice, we use "$\subset_e$" to denote the end-extension relation.) Let $V(w, x, y)$ be a Σ_1^0 $\mathscr{L}$-formula expressing, *in* ω, the relation $y \simeq \phi_w(x)$ where ϕ_w is the w-th unary partial recursive function (in some standard indexing of all such functions). By Lemma 5.1, we are at liberty to assume that $V(w, x, y)$ is *provably univalent*, i.e., that

$$PA \vdash \forall w \forall x \forall y \forall z[(V(w, x, y) \wedge V(w, x, z)) \rightarrow y = z].$$

A careful examination of the proof of Theorem 2.6 in [H] shows that there exists an element b of $\mathrm{N}(X) - \omega$ such that the formula $\forall x \leqq b \exists y[y < z \wedge V(y, b, x)]$ is a parametric definition of $\mathrm{N}(X) - \omega$ in $\mathrm{N}(X)$. Let ψ be the sentence $\forall w \forall x \forall y \forall z[(V(w, x, y) \wedge V(w, x, z)) \rightarrow y = z]$. Then $\mathcal{M} \models \psi$, since $PA \vdash \psi$ and ψ is Π_1^0. It follows, since $\mathrm{N}(X) \subset_e \mathcal{M}$, that the interval $\{z \mid \omega < z \leqq b\}$ is defined, *in* $\mathcal{M}$, by the parametric formula $z \leqq b \wedge \forall x \leqq b \exists y[y < z \wedge V(y, b, x)]$. (To see this, we first note that if z is an element of $\mathcal{M}$ for which $\omega < z \leqq b$, then, since $\mathrm{N}(X) \models \forall x \leqq b \exists y[y < z \wedge V(y, b, x)]$ and $\mathrm{N}(X) \subset_e \mathcal{M}$, we have $\mathcal{M} \models z \leqq b \wedge \forall x \leqq b \exists y[y < z \wedge V(y, b, x)]$. In the other direction, suppose z is an element of $\mathcal{M}$ such that $\mathcal{M} \models z \leqq b \wedge \forall x \leqq b \exists y[y < z \wedge V(y, b, x)]$. Then, since V is univalent in $\mathcal{M}$ and there are infinitely many $x \leqq b$ in $\mathcal{M}$, we conclude that $z \notin \omega$.) This, however, is not quite the definition of $\{z \mid \omega < z \leqq b\}$ in $\mathcal{M}$ that we want: it has an awkwardly-placed universal quantifier. Let $V(y, b, x)$ be of the form $\exists \mathbf{w} R(\mathbf{w}, y, b, x)$, where R is bounded. We note, to begin with, that it is all right to assume that the sequence $\mathbf{w} = w_1, \ldots, w_k$ is of length 1; for, the equivalence

$$\forall y \forall z \forall x[\exists w_1 \cdots \exists w_k R(w_1, \ldots, w_k, y, z, x)$$
$$\leftrightarrow \exists w \exists w_1 < w \cdots \exists w_k < w R(w_1, \ldots, w_k, y, z, x)]$$

is clearly true in all models of PA_3 (indeed, PA_2) and so, in particular, in $\mathcal{M}$. Similarly, we have:

$$\mathcal{M} \models \forall z \forall t \forall u$$
$$\leqq t[\exists y(y < z \wedge \exists w \exists w_1 < w \cdots \exists w_k < w R(w_1, \ldots, w_k, y, t, u))$$
$$\leftrightarrow \exists w \exists y < z \exists w_1 < w \cdots \exists w_k < w R(w_1, \ldots, w_k, y, t, u)].$$

Thus, $\{z \mid \omega < z \leqq b\}$ is defined in $\mathcal{M}$ by the parametric formula $\tau(z)$, where $\tau(z)$ is

$$z \leqq b \wedge \forall x \leqq b \exists w \exists y < z \exists w_1 < w \cdots \exists w_k < w R(w_1, \ldots, w_k, y, b, x).$$

Now, each instance of the Collection Schema $B\Sigma_0$ (see [P]) is logically equivalent to a sentence in Π_3^0 form (since each such instance has unbounded quantifier structure $\forall[\forall \exists \rightarrow \exists]$). Moreover, the converse implication for any instance of $B\,\Sigma_0$ is logically equivalent to a sentence in Π_3^0 form. Applying these facts to $\tau(z)$, we see that $\{z \mid \omega < z \leqq b\}$ is defined in $\mathcal{M}$ by the parametric Σ_1^0 formula

$$z \leqq b \wedge \exists w \forall x \leqq b \exists t < w \exists y < z \exists w_1 < t \cdots \exists w_k < tR(w_1, \ldots, w_k, y, b, x).$$

Now we are in a position to clinch matters. We observe that each instance of the Least Element Schema $L\,\Sigma_1$ (see [P]) is logically equivalent to a sentence in Π_3^0 form and so holds in $\mathcal{M}$; then, applying $L\,\Sigma_1$ to the above parametric Σ_1^0 formula, we conclude that, in $\mathcal{M}$, the set $\{z \mid \omega < z \leqq b\}$ has a least element, which is of course absurd. Our theorem follows.

To Theorem 5.2 we add the following postscript: it is rather easy to produce non-standard mocels of full arithmetic that are end-extendible to Nerode semirings. Suppose $\mathcal{N}$ and $\mathcal{M}$ are countable $\mathcal{L}$-structures with $\mathcal{N} \subseteq \mathcal{M}$. We denote by "Co $f_{\mathcal{M}}\mathcal{N}$" the set $\{x \in |\mathcal{M}| \mid \exists y \in \mathcal{N}(x \leqq y)\}$.

5.3. Proposition. *Suppose $\mathcal{N}$, $\mathcal{M}$ are (non-standard) 2-correct $\mathcal{L}$-structures, with $\mathcal{N} \subseteq \mathcal{M}$. Then* Co $f_{\mathcal{M}}\mathcal{N}$ *is also a 2-correct $\mathcal{L}$-structure.*

Proof. By [H, Th. 1.8], it suffices to show that Co $f_{\mathcal{M}}\mathcal{N}$ is closed under all $\mathcal{M}$-recursive functions. So let $f(x_1, \ldots, x_n)$ be $\mathcal{M}$-recursive; and let $a_1, \ldots, a_n \in \mathrm{Co}\, f_{\mathcal{M}}\mathcal{N}$. Let $b_1 \geqq a_1, \ldots, b_n \geqq a_n$, with $b_i \in |\mathcal{N}|$. Then $2^{b_1} \cdots p_n^{b_n}$ is an element of $|\mathcal{N}|$ that simultaneously bounds all of $a_1, \ldots, a_n$. Let $g_f(x) = \Sigma_{x_1 \leqq x, \ldots, x_n \leqq x} f(x_1, \ldots, x_n)$. Then g_f is monotone nondecreasing on ω, and the assertion that it has that property can be written as a 2-quantifier truth about ω. Hence, g_f is monotone nondecreasing as an $\mathcal{M}$- (or $\mathcal{N}$-) recursive function. But also, the sentence

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n[(x_1 \leqq y_1 \wedge \cdots \wedge x_n \leqq y_n) \rightarrow f(x_1, \ldots, x_n) \leqq g_f(2^{y_1} \cdots p_n^{y_n})]$$

is a 2-quantifier truth in ω, and so holds in both $\mathcal{M}$ and $\mathcal{N}$. Thus: in $\mathcal{M}$, we have that $f(a_1, \ldots, a_n) \leqq g_f(2^{b_1} \cdots p_n^{b_n})$; and $g_f(2^{b_1} \cdots p_n^{b_n}) \in |\mathcal{N}|$ since $\mathcal{N}$ is closed under all $\mathcal{M}$-recursive functions. Thus $f(a_1, \ldots, a_n) \in \mathrm{CO}\, f_{\mathcal{M}}\mathcal{N}$, and the proof is complete.

5.4. Theorem. *Any Nerode semiring that embeds a non-standard correct $\mathcal{L}$-structure is in fact an end-extension of a non-standard correct $\mathcal{L}$-structure.*

Proof. Let $\mathcal{M}_0$ be a countable correct $\mathcal{L}$-structure. Applying [McL$_2$, Th.

1.1] let N(X) be a Nerode semiring such that $\mathcal{M}_0 \subseteq \mathrm{N}(X)$. By Proposition 5.3, $\mathcal{M} = \mathrm{Co}\, f_{\mathrm{N}(X)}\mathcal{M}_0$ is 2-correct; since $\mathcal{M}_0$ is cofinal in $\mathcal{M}$, it follows from [G, Th. 3] and Lemma 3.3 that $\mathcal{M}$ is correct. Obviously, N(X) cofinally extends $\mathcal{M}$; that it does so *properly* is a trivial consequence of any number of things, including [F–S–T].

All of this leaves a couple of rather obvious questions, one of which we shall record in §6.

REMARKS

(1) Among other things, Theorem 5.2 provides a certain strengthening of the classical Feferman–Scott–Tennenbaum theorem [F–S–T].

(2) An analysis of the proof of 5.2 in the light of [P, Prop. 1] shows that N(X) cannot be end-extended to any $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models P^- +$ bounded induction $+ L\,\Sigma_1$.

(3) Our proof of Theorem 5.2 is, as should be evident, the result of thinking about Hirschfeld's proof of [H, Th. 2.6]. A result closely akin to [H, Th. 2.6] is [H–W, Th. 8.29]; and in fact the details of our proof of Theorem 5.2 are to the specific end of replacing a predicate of the kind used in [H, proof of Th. 2.6] by one of the type appearing in [H–W, Th. 8.29]. Note, however, that the class of structures with which [H–W, Th. 8.29] is concerned is *disjoint from* the class of models of PA_3.

(4) We can't apply [H–W, Th. 8.29] directly to N(X), to shorten our proof of Theorem 5.2. This is because (see [H–W, Ch. 9]) not all Nerode semirings are existentially complete for the class of correct models; some are, some aren't.

6. Open problems

Sections 3, 4, and 5 all suggest some problems, two of these having previously been mentioned in [McL_3].

Re §3:

3a. Can we replace *"PA_4"* by *"PA_3"* in Theorem 3.4?

3b. Are all cofinal embedders minimal?

Re §4:

4a. Do there, in fact, exist any "mixed models" among the $\forall\exists$-correct subsemirings of Λ; i.e., are there any $\forall\exists$-correct subsemirings $\mathbf{R}$ of Λ

such that **R** contains both nonregressive isols and infinite *regressive* isols?

4b. (cf. [McL_3, §6]. Is every countable $\forall\exists$-correct semiring of isols *included in* (rather than merely isomorphic to a subsemiring of) some Nerode semiring?

4c. (cf. [McL_3, §6]). Do there exist any *uncountable* $\forall\exists$-correct subsemirings of Λ?

Re §5: Can one Nerode semiring properly end-extend another? (This is equivalent to asking: can some Nerode semiring be properly end-extended by a 2-correct $\mathscr{L}$-structure?)

Finally, we pose a question that is not directly tied to any particular section of the present paper, but is simply a matter of "general considerations." In [N_3, Th. 5.3], Nerode proved the existence of countable correct subsemirings **R** of Λ with the following property: for each Σ_1^0 $\mathscr{L}$-formula $\phi(x_1, \ldots, x_k)$ defining an r.e. relation $R \subseteq \omega^k$, and for every $X \in |\mathbf{R}|^k$,

$$\mathbf{R} \models \phi(X) \Leftrightarrow X \in R_\Lambda.$$

(Indeed, Nerode showed that **R** can be an isolic copy of *any* countable correct model.) We ask: does the above equivalence hold true *in Nerode semirings* (i.e., upon replacing **R** by $\mathrm{N}(X)$)?

(The left-to-right implication does indeed hold in $\mathrm{N}(X)$, because of *DPRM* and the basic Nerode metatheory; the other direction is not so clear.)

Remarks added in proof

(1) A quicker proof of a better version of Theorem 3.2 is available. Namely, by combining Theorem 5.2 and Proposition 5.3, we can easily verify that *every* Nerode semiring is isomorphic to a proper cofinal subsemiring of some other Nerode semiring.

(2) The statement of Theorem 5.4 should be improved to read: "Let T be a diophantine correct complete extension of PA, and let $\mathrm{N}(X)$ be any Nerode semiring that embeds a nonstandard model of T. Then $\mathrm{N}(X)$ end-extends a nonstandard model of T." No appeal to Lemma 3.3 is needed for the proof of this; [G] alone clearly suffices.

(3) We have now observed that the answer to the question about §5 is "no." In fact, we can prove: No Nerode semiring can be properly end-extended by a model of PA_2.

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